EQUIVARIANT CLOSURE OPERATORS AND TRISP CLOSURE MAPS

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Abstract. A trisp closure map ϕ is a special map on the vertices of a trisp T with the property that T collapses onto the subtrisp induced by the image of ϕ. We study the interaction between trisp closure maps and group operations on the trisp, and give conditions such that the quotient map is again a trisp closure map. Special attention is on the case that the trisp is the nerve of an acyclic category, and the relationship between trisp closure maps and closure operators on posets is studied.

1. Introduction

This paper aims to bring together the subjects of two different papers: [BK05], where Babson and Kozlov studied quotients of posets and conditions under which these commute with the nerve functor; and [Ko08b], where Kozlov introduced trisp closure maps, which are a compact certificate for the collapsibility of a trisp to a certain subtrisp.

Section 2 gives definitions for the needed objects: acyclic categories and their nerves, quotients of trisps, quotients of acyclic categories. Trisp closure maps get introduced in Section 3, and there is a discussion on the relationship between them and closure operators on posets. The main results of this paper can be found in Section 4: Conditions that are sufficient to guarantee the regularity of a quotient trisp also guarantee that the quotient of a trisp closure map is a closure map on the quotient trisp. In particular, these conditions are always fulfilled if the trisp is the nerve of an acyclic category, with the group operation induced by an operation on the acyclic category. Conversely, a trisp closure map on a quotient trisp can be lifted whenever the lifted map can be defined in a natural way and the original trisp is actually a simplicial complex. In Section 5, we shed more light on the connection between closure operators and trisp closure maps. We now consider the quotient of a poset, equipped with a closure operator. As this quotient is taken in the category of acyclic categories, we cannot expect the quotient of the closure operator to be a closure operator again. But we do still obtain a trisp closure map on the nerve of the poset quotient.

Finally, we apply our results in Section 6 to the $\mathcal{S}_n$-action on the barycentric subdivision of the complex of disconnected graphs introduced by Vassiliev.

2. Definitions

2.1. **Acyclic categories and posets.** In this paper, let C be a finite acyclic category, that is, a finite category where only identity morphisms have inverses. That means that C can be pictured with all arrows pointing upward; we write $s(m)$ for the source object of $m \in \mathcal{M}(C)$, $t(m)$ for the target object. We write $x \parallel y$, if $\mathcal{M}(x, y) = \mathcal{M}(y, x) = \emptyset$ for two objects x, y. By P we always denote a

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finite poset, which we understand here as an acyclic category where for any two objects x, y there is at most one morphism $x \to y$, written $x \leq y$. An AC-map on C is a functor $\phi : C \to C$; if C is actually a poset, then this notion coincides with that of an order-preserving map.

Let G be a group acting on C. An action of G on C is a functor $\mathcal{A}_G$ from the category G to the category of acyclic categories. This means that each group element g acts as an automorphism $\mathcal{A}_g$ on C; we write ga instead of $\mathcal{A}_g a$ for some morphism or object a. In particular, such a group action here is always horizontal, meaning that $gx \neq x$ implies $gx \parallel x$ for all objects x. An AC-map ϕ on C is G-equivariant if it commutes with $\mathcal{A}_g$ for all group elements g.

2.2. **Nerves of acyclic categories.** For the definition of a trisp, see for example [Ko08a, Ch. 2]; basically it is a generalization of an abstract simplicial complex, where on the one hand there may be multiple simplices with the same vertex set, and on the other hand there is an order on the simplices compatible with taking boundaries. A trisp is regular if the number of distinct vertices of each simplex equals the dimension of the simplex plus 1. To each acyclic category C, we can associate a trisp $\Delta(C)$, called the nerve of C. The 0-simplices (or vertices) of $\Delta(C)$ are the objects of C; the t-simplices are chains of t composable non-identity morphisms, e.g. $\sigma = a_0 \stackrel{m_1}{\to} a_1 \stackrel{m_2}{\to} a_2 \to \ldots \stackrel{m_t}{\to} a_t$. The boundary simplices of σ are as follows: $\partial_0\sigma = a_1 \stackrel{m_2}{\to} a_2 \to \ldots \stackrel{m_t}{\to} a_t$, $\partial_i\sigma = a_0 \stackrel{m_1}{\to} \ldots \stackrel{m_{i-1}}{\to} a_{i-1} \stackrel{m_{i+1}\circ m_i}{\to} a_{i+1} \to \ldots \stackrel{m_t}{\to} a_t$, $\partial_t\sigma = a_0 \stackrel{m_1}{\to} a_1 \stackrel{m_2}{\to} a_2 \to \ldots \stackrel{m_{t-1}}{\to} a_{t-1}$. Thus, the minimal vertex of σ is a_0. The nerve of any acyclic category is a regular trisp, so in the following we will be concerned only with regular trisps without explicit mention.

Even more specially, nerves of acyclic categories are flag complexes. That is, they are maximal under the condition that 1-skeleta of simplices are unique. Thus, the trisp as a whole is uniquely determined by its 1-skeleton.

There is an induced action of G on $\Delta(C)$, by [Ko08a, Prop. 14.4] $\Delta(C)/G$ is again a regular trisp and the simplices are exactly the orbits of simplices of $\Delta(C)$, e.g. $G\sigma = G(a_0 \stackrel{m_1}{\to} a_1 \stackrel{m_2}{\to} a_2 \to \ldots \stackrel{m_t}{\to} a_t)$ and the boundary maps work as expected: $\partial_i(G\sigma) = G(\partial_i\sigma)$.

2.3. **The quotient** C/G**.** Following [BK05], we define the quotient C/G of C by its G-action as the colimit of the functor $\mathcal{A}_G$. In our finite case here one can give an explicit description of C/G. The objects $[a]$ are simply the G-orbits of objects a of C. The morphisms $[x]$ are equivalence classes of morphisms of C, where the equivalence relation is induced by the G-action and composition, with the transitive closure taken. Stated precisely, we have $[x] = [y]$ for $x, y \in \mathcal{M}(C)$ if there exist $z_1, z_2, \ldots, z_n \in \mathcal{M}(C)$, $z_1 = x, z_n = y$, and decompositions $z_i = z^+_{i,t_i} \circ z^+_{i,t_i-1} \circ \ldots \circ z^+_{i,1}$ for $i = 1, 2, \ldots, n-1$ and $z_i = z^-_{i,t_{i-1}} \circ z^-_{i,t_{i-1}-1} \circ \ldots \circ z^-_{i,1}$ for $i = 2, 3, \ldots, n$, such that $Gz^+_{i,j} = Gz^-_{i+1,j}$ for all appropriate i, j.

As C/G is again an acyclic category, we can consider its nerve $\Delta(C/G)$. The universal property of colimits guarantees the existence of a canonical map $\lambda : \Delta(C)/G \to \Delta(C/G)$, with $\lambda(G(a_0 \stackrel{m_1}{\to} a_1 \stackrel{m_2}{\to} a_2 \to \ldots \stackrel{m_t}{\to} a_t)) = ([a_0] \stackrel{[m_1]}{\to} [a_1] \stackrel{[m_2]}{\to} [a_2] \to \ldots \stackrel{[m_t]}{\to} [a_t])$. On the 0-skeleta, λ is an isomorphism, as $[a] = Ga$ are the vertices of $\Delta(C/G)$ and $\Delta(C)/G$, respectively. Necessary and sufficient conditions for this map to be an isomorphism have been studied in [BK05]; in particular λ is always surjective. As we will make heavy use of this fact, the proof is repeated here.

Proposition 1. [BK05, Prop. 3.1] *Let C be an acyclic category with a G-action. Then the canonical map $\lambda : \Delta(C)/G \to \Delta(C/G)$ is surjective.*

Proof. Let $[a_0] \stackrel{[m_1]}{\to} [a_1] \stackrel{[m_2]}{\to} [a_2] \to \ldots \stackrel{[m_t]}{\to} [a_t]$ be a simplex of $\Delta(C/G)$. We will construct a λ-preimage inductively. For $t = 0$ the only possible choice as mentioned above is Ga_0. If we have found $b_0 \stackrel{n_1}{\to} b_1 \stackrel{n_2}{\to} \ldots \stackrel{n_{t-1}}{\to} b_{t-1}$ with $[n_i] = [m_i]$ for all $i = 1, \ldots, t-1$, implying $b_{t-1} \in [a_{t-1}]$, there exists $g \in G$ such that $b_{t-1} = g \cdot s(m_t)$. Thus choosing $n_t = g \cdot m_t$ yields an extension to the composable morphism chain, with $t(n_t) \in t([m_t]) = [a_t]$. □

3. Trisp closure maps

Definition 2. A *closure operator* on a poset P is an order-preserving map $\phi : P \to P$ with $\phi^2 = \phi$. It is a descending (ascending) closure operator, if $\phi(x) \leq x$ ($\phi(x) \geq x$) for all $x \in P$.

It is a well-known fact that a monotone closure operator ϕ on P induces a strong deformation retract from the order complex of P to the order complex of $\phi(P)$; see e.g. [Bj96, Cor. 10.12]. Even more, $\Delta(P)$ actually collapses onto $\Delta(\phi(P))$ (see [Ko04, Th. 2.1]). Forgetting about the underlying poset and just considering a trisp, we arrive at the central definition of this paper, due to Kozlov ([Ko08b]):

Definition 3. A *trisp closure map* on a trisp T is a partition of the vertex set of T into the *blue* vertices B and the *red* vertices R, together with a map $\phi : B \to R$ with the following property: Let σ be a simplex of T containing at least one blue vertex; let b be the minimal blue vertex of σ. Then either

- $\phi(b)$ is a vertex of σ, and removing $\phi(b)$ yields another (unique) simplex of T
- or $\phi(b)$ is not a vertex of σ, then there exists a unique vertex τ of T that contains $\phi(b)$ as a vertex and σ as a boundary simplex of codimension 1.

Replacing "minimal" with "maximal" yields no conceptual difference, and all statements in this paper which do not explicitely mention the choice made still hold.

This definition is made worthwile by the following theorem due to Kozlov.

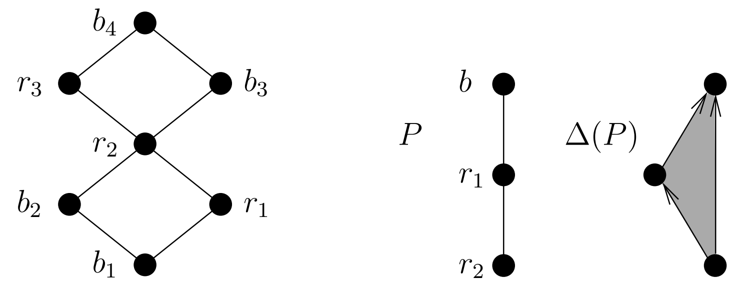
Theorem 4. [Ko08b, Thm. 2.2] *Let T be a regular trisp with a trisp closure map $\phi : B \to R$. Then T collapses on the subtrisp T_R, consisting of those simplices of T that contain only red vertices.*

Remark 5. In particular, any descending (ascending) closure operator ϕ on a poset P induces a trisp closure map $\bar{\phi}$ on $\Delta(P)$ with minimal (maximal) vertices chosen, by setting $R = \phi(P), B = P \backslash R, \bar{\phi} = \phi_{|B}$, implying that $\Delta(P)$ collapses onto $\Delta(\phi(P))$ ([Ko08b, Cor. 2.5]). If P carries a G-action and ϕ is G-equivariant, then R and B are closed under G, since with $r = \phi(p) \in R$ also $gr = g\phi(p) = \phi(gp)$ is in R for all group elements g.

The relationship between trisp closure maps and poset closure operators is not as close as it appears on the first glance though. For example, Figure 3.1a gives a closure operator on a poset that does not induce a trisp closure map: The simplex $(b_1 < b_2 < b_3 < b_4)$ cannot be extended by r_3, and the situation is not remedied by considering the dual poset instead. On the other hand, Figure 3.1b gives an example of a trisp closure map that is not induced by any order-preserving map.

If we shift our point of view from posets to acyclic categories, as arises naturally when asking about a trisp closure map on $\Delta(P/G)$, we loose this tool. But we can still ask for any AC-map $\phi : C \to C$ whether the induced map on the vertices of $\Delta(C)$ is a trisp closure map for some choice of B. Necessary conditions are:

Condition 6. B and $\phi(B)$ are disjoint, and $|\mathcal{M}(x, \phi(x))| + |\mathcal{M}(\phi(x), x)| = 1$ for all $x \in B$.



(a) $\phi : b_1 \mapsto r_1, b_2 \mapsto r_2, b_3 \mapsto r_2, b_4 \mapsto r_3$

(b) $\bar{\phi} : b \mapsto r_2$

Figure 3.1:

4. Closure maps on $\Delta(C)/G$

Consider first some arbitrary trisp T with a G-action, where G is a finite group. According to Kozlov [Ko08a, Section 14.1.2], the following condition ensures that T/G is again a regular trisp.

Condition (R)**.** For any $g \in G$ and any simplex σ of T, the simplex $g(\sigma) \cap \sigma$ is fixed pointwise by g.

It turns out that this condition ensures that the quotient map of a G-equivariant trisp closure map on T is a trisp closure map on T/G.

Proposition 7. *Let T be a finite trisp with a G-action fulfilling Condition R. Let $\phi : B \to R$ be a G-equivariant trisp closure map on T, where B and R are closed under G (that is, $G \cdot B \cap G \cdot R = \emptyset$). Then the quotient map ϕ_G is a trisp closure map on T/G.*

Proof. Let $G\sigma$ be a simplex of T/G, with a minimal blue vertex Gb. Choose the representative b such that it is a vertex of σ; since the group action respects trisp order, b is the minimal blue vertex of σ as well. There is only something to prove if $\phi_G(Gb) = G\phi(b)$ is not a vertex of $G\sigma$. Then $\phi(b)$ is not a vertex of σ and there exists a unique extension simplex τ of σ by $\phi(b)$; that is, $\phi(b)$ is the j-th vertex of τ and $\partial_j\tau = \sigma$. Then $\partial_j(G\tau) = G\sigma$ and $G\phi(b)$ is a vertex of $G\tau$, so $G\tau$ extends $G\sigma$.

Assume that there is another extension $G\tau'$ of $G\sigma$. Choose the representative τ' such that $\partial_j\tau' = \sigma$ for some j, then τ' contains the vertices b and $g\phi(b)$, where $g \in G$ such that $g\phi(b)$ is the representative of $G\phi(b)$ in τ'. Thus there is a simplex ρ in T with vertex set $\{b, g\phi(b)\}$. Assume that $g\phi(b) \neq \phi(b)$, since otherwise $\tau' = \tau$. As $\phi(b)$ is a red vertex and thus $g\phi(b)$ is red as well, there must exist an extension simplex of ρ, with vertices $b, \phi(b), g\phi(b)$, and a boundary simplex ρ' with vertices $\phi(b)$ and $g\phi(b)$. Then $g\phi(b) \in g\rho' \cap \rho'$, so by Condition S we have $g\phi(b) = g^2\phi(b)$ and thus $\phi(b) = g\phi(b)$. Hence $\tau = \tau'$ and thus the extension simplex of $G\sigma$ is unique in T/G. $\square$

As shown in [Ko08a, Prop. 14.4], Condition R is automatically fulfilled for $T = \Delta(C)$ for some acyclic category C, where the group action on T is induced by a group action on C. So we obtain the following corollary.

Corollary 8. *Let C be a finite acyclic category with G-action. Let $\phi : C \to C$ be a G-equivariant AC-map that induces a trisp closure map $\bar{\phi} : B \to R$ on $\Delta(C)$, where B and R are closed under G (that is, $G \cdot B \cap G \cdot R = \emptyset$). Then the quotient map $\bar{\phi}_G$ is a trisp closure map on $\Delta(C)/G$.*

Now we consider the opposite direction: Given a trisp T with a group action G fulfilling Condition R (as we want to have regularity of T/G ensured), and a trisp closure map $\psi : \bar{B} \to \bar{R}$ on T/G, we wish to obtain a trisp closure map $\phi : B \to R$ on T such that $\phi_G = \psi$. There are no choices for $B = \{b \in X | X \in \bar{B}\}$ and $R = \{r \in X | X \in \bar{R}\}$.

Condition (C)**.** For each $b \in B$ there is a unique vertex $r_b \in \psi(Gb)$ such that there exists a simplex with vertex set $\{b, r_b\}$ and this simplex is unique as well.

In other words, if Condition C is fulfilled, then ϕ is well-defined by setting $\phi(b) = r_b$.

Proposition 9. *Condition C is necessary for the existence of a trisp closure map $\phi : B \to R$ on T with $\phi_G = \psi$.*

Proof. For each $b \in B$ the image $\phi(b) =: r$ must be chosen from $\psi(Gb)$. As ψ is a closure map, there exists exactly one simplex $G\sigma$ in T/G with vertex set $\{Gb, \psi(Gb)\}$, and a representative σ can be chosen with b as a vertex. The other vertex of σ is then r. If there exists another candidate $gr \in \psi(Gb)$ such that there is a simplex σ' with vertex set $\{b, gr\}$, then an extension τ of σ' must exist, with vertex set $\{b, gr, \phi(b) = r\}$. One boundary simplex τ' of τ has vertex set $\{gr, r\}$, which by the route of $g\tau' \cap \tau' \ni gr$ and Condition R implies $gr = r$.

If there are two different simplices σ, σ' with vertex set $\{b, r\}$, then both extend the simplex b, in contradiction to ϕ being a trisp closure map. $\square$

In general, Condition C is not sufficient to obtain ϕ. Consider for example the following regular trisp: 0-simplices are b, x, r; 1-simplices are $(b, x), (b, r), (x, r)$ and two 2-simplices σ and τ, both having all the 1-simplices as boundaries (a filled triangle with double filling). $\mathbb{Z}_2$ acts on this trisp: the nonidentity element interchanges σ and τ, leaving all other simplices fixed. This action is a trisp action fulfilling Condition C. The quotient trisp is just a filled triangle, and $\psi : \{\mathbb{Z}_2 b\} \to \{\mathbb{Z}_2 x, \mathbb{Z}_2 r\}$, mapping $\mathbb{Z}_2 b$ to $\mathbb{Z}_2 r$, is a $\mathbb{Z}_2$-equivariant trisp closure map that is not the quotient of any trisp closure map in the original trisp.

On the other hand, if T is an abstract simplicial complex (meaning here that different simplices have different vertex sets), then Condition C is indeed sufficient.

Proposition 10. *Let T be an abstract simplicial complex with a group G acting on T under Condition R; let $\psi : \bar{B} \to \bar{R}$ be a trisp closure map on T/G. Then the following are equivalent:*

(1) ψ fulfills Condition C

(2) $\phi : B \to R$ is a trisp closure map on T with $\phi_G = \psi$, where $B = \{b \in X | X \in \bar{B}\}$ and $R = \{r \in X | X \in \bar{R}\}$ and $\phi(b) = r_b$, with r_b as in Condition C.

Proof. Condition C is necessary by Proposition 9. To show that it is sufficient, consider a simplex σ of T, with minimal blue vertex b and $\phi(b) \notin \sigma$. Then Gb is the minimal blue vertex of $G\sigma$ and there exists an extension $G\tau$ of $G\sigma$. So j exists with $G\sigma = \partial_j(G\tau) = G(\partial_j \tau)$. Choose a representative τ such that $\partial_j \tau = \sigma$, which is possible by the regularity of the G-action on T implied by Condition R. Then the j-th vertex v_j of τ is in $\psi(Gb)$, and there is a subsimplex $\{b, v_j\}$ of τ, so by Condition C we have $v_j = \phi(b)$. Thus τ is the unique extension simplex of σ. $\square$

In particular, the nerve of a poset P is always an abstract simplicial complex, and a group G acting on P induces an action on $\Delta(P)$ that always fulfills Condition R.

$$\begin{tikzcd}
a = a^+_{1,t_1} \arrow[d, dash] \arrow[dr, phantom, "\xrightarrow{g_{1,t_1-1}}"] & a^-_{2,t_1} \arrow[d, dash] & = & a^+_{2,t_2} \arrow[d, dash] \arrow[dr, phantom, "\xrightarrow{g_{2,t_2-1}}"] & a^-_{3,t_2} = a \arrow[d, dash] \\
a^+_{1,t_1-1} \arrow[d, dash] \arrow[dr, phantom, "\xrightarrow{g_{1,t_1-2}}"] & a^-_{2,t_1-1} \arrow[d, dash] & & a^+_{2,t_2-1} \arrow[d, dash] \arrow[dr, phantom, "\xrightarrow{g_{2,t_2-2}}"] & a^-_{3,t_2-1} \arrow[d, dash] \\
\vdots \arrow[d, dash] \arrow[dr, phantom, "\xrightarrow{g_{1,1}}"] & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] \arrow[dr, phantom, "\xrightarrow{g_{2,1}}"] & \vdots \arrow[d, dash] \\
b = a^+_{1,1} & a^-_{2,1} & = & a^+_{2,1} & a^-_{3,1} = c
\end{tikzcd}$$

Figure 5.1: Example situation with $n = 3$

5. CLOSURE MAPS ON $\Delta(P/G)$

As P/G is usually not a poset, we cannot hope that the quotient of a closure operator is again a closure operator. But it turns out that the induced map on $\Delta(P/G)$ is still a trisp closure map.

All results in this section hold for ascending closure operators as well, using the "maximal" version of trisp closure maps.

Lemma 11. *Let $\phi : P \to P$ be a G-equivariant descending closure operator. Then $[b < a] = [c < a]$ in P/G implies that $[\phi(b) < \phi(a)] = [\phi(c) < \phi(a)]$ in P/G and also in $\phi(P)/G$.*

Proof. Long version: Writing this out as in Section 2.3, this means that there exist $n, t_1, \ldots, t_n \in \mathcal{N}, a^+_{ij}, a^-_{ij} \in P, g_{ij} \in G$ such that $b = a^+_{11} < a^+_{12} < \ldots < a^+_{1t_1} = a$, $a^-_{i1} < a^-_{i2} < \ldots < a^-_{it_{i-1}}$, $a^+_{i1} < a^+_{i2} < \ldots < a^+_{it_i}$, $c = a^-_{n1} < a^-_{n2} < \ldots < a^-_{nt_{n-1}} = a$, with $a^-_{i1} = a^+_{i1}$, $a^-_{it_{i-1}} = a^+_{it_i}$, $g_{ij}a^+_{ij} = a^-_{i+1,j}$, $g_{ij}a^+_{i,j+1} = a^-_{i+1,j+1}$.

An example of the situation is shown in Figure 5.1.

We will prove the statement by induction on n.

If $n = 2$, then by applying ϕ to the chains, we obtain $\phi(b) = \phi(a^+_{11}) \leq \phi(a^+_{12}) \leq \ldots \leq \phi(a^+_{1t_1}) = \phi(a)$, $\phi(c) = \phi(a^-_{21}) \leq \phi(a^-_{22}) \leq \ldots \leq \phi(a^-_{2t_1}) = \phi(a)$. In each of these, we might get subsequences with equality $\phi(a_{jk}) = \phi(a_{j,k+1}) = \ldots = \phi(a_{j,k+l})$. Reduce the index set $\{1, 2, \ldots, t_1\}$ by keeping only the first index in each of these sequences; this yields the same result for both chain images because the G-action is horizontal. To simplify notation, denote the new index set with $\{1, 2, \ldots, t_1\}$ as well. Thus we get chains $\phi(b) = \phi(a^+_{11}) < \phi(a^+_{12}) < \ldots < \phi(a^+_{1t_1}) = \phi(a)$, $\phi(c) = \phi(a^-_{21}) < \phi(a^-_{22}) < \ldots < \phi(a^-_{2t_1}) = \phi(a)$, with $g_{ij}\phi(a^+_{ij}) = \phi(g_{ij}a^+_{ij}) = \phi(a^-_{i+1,j})$, $g_{ij}\phi(a^+_{i,j+1}) = \phi(a^-_{i+1,j+1})$. We conclude that $[\phi(b) < \phi(a)] = [\phi(c) < \phi(a)]$.

Assume that $[\phi(b) < \phi(a)] = [\phi(a^-_{n-1,1}) < \phi(a^-_{n-1,t_{n-2}})]$. Proceeding as above using the chains $a^+_{n-1,1} < a < \ldots < a^+_{n-1,t_{n-1}}$, $c = a^-_{n1} < a^-_{n2} < \ldots < a^-_{nt_{n-1}} = a$ yields $[\phi(c) < \phi(a)] = [\phi(a^+_{n-1,1}) < \phi(a^+_{n-1,t_{n-1}})] = [\phi(a^-_{n-1,1}) < \phi(a^-_{n-1,t_{n-2}})] = [\phi(b) < \phi(a)]$. $\square$

Proposition 12. *Let $\phi : P \to P$ be a G-equivariant descending closure operator. Then $\Delta_R(P/G)$, the subtrisp of $\Delta(P/G)$ induced by the vertex set $R = G \cdot \phi(P)$, equals $\Delta(\phi(P)/G)$.*

Proof. The vertex sets of both trisps coincide by definition. By Lemma 11, the 1-skeleta coincide as well, and because both trisps are flag complexes, this is sufficient to guarantee equality. □

Proposition 13. *Let $\phi : P \to P$ be a G-equivariant descending closure operator. Then the induced map ϕ_G is a trisp closure map on $\Delta(P/G)$.*

Proof. Let $R = \phi_G(P/G)$ be the red vertices of P/G, $B = (P/G)\backslash R$ be the blue vertices. Consider some simplex σ containing a minimal blue vertex $[b]$ and not containing $\phi_G([b])$. There is a λ-preimage $G(a_1 < a_2 < \ldots < a_r)$ of σ, which can be extended by $[\phi(b)] = \phi_G([b])$, since the induced map on $\Delta(P)/G$ is a trisp closure map by Remark 5 and Proposition 7. Taking the λ-image of this extension provides an extension of σ in $\Delta(P/G)$.

The minimal blue vertex in σ is $[b] = [a_i]$ and ϕ is descending, so there can be no subsimplex $([\phi(b)] \stackrel{[x]}{\to} [a_{i-1}] \stackrel{[m_{i-1}]}{\to} [b])$ of an extension of σ. Otherwise we could obtain a preimage $g\phi(b) < a_{i-1} < b$ by properly choosing representatives $b, a_{i+1}, g\phi(b)$. Applying ϕ leads to $g\phi(b) \leq \phi(a_{i+1}) = a_{i+1} \leq \phi(b)$, thus $\phi(b) = a_{i+1} = g\phi(b)$. Therefore any extension of σ must be of the form $([a_1] \stackrel{[m_1]}{\to} [a_2] \stackrel{[m_2]}{\to} \ldots \stackrel{[m_{i-2}]}{\to} [a_{i-1}] \stackrel{[y]}{\to} [\phi(b)] \stackrel{[x]}{\to} [b] = [a_i] \stackrel{[m_i]}{\to} \ldots \stackrel{[m_{r-1}]}{\to} [a_r])$.

The only remaining question is whether there are different extensions to σ. These can vary only in their choice of $[x]$ and $[y]$, under the condition that $[x \circ y] = [m_{i-1}]$. So we only need to consider the situation where $[r] \stackrel{[y]}{\to} [\phi(b)] \stackrel{[x]}{\to} [b]$ with $[r] \in R$, and $[r] \stackrel{[y']}{\to} [\phi(b)] \stackrel{[x']}{\to} [b]$ with $[x' \circ y'] = [x \circ y]$ and prove that then $[x] = [x']$ and $[y] = [y']$.

(1) Let $g \in G$ such that $[x] = [g\phi(b) < b]$. Since $g\phi(b) = \phi(g\phi(b)) \leq \phi(b)$, we see that $[x] = [\phi(b) < b]$; by the same argument $[x'] = [\phi(b) < b]$ holds.

(2) Choose a representative r such that $[y] = [r < \phi(b)]$, let $g \in G$ such that $[y'] = [gr < \phi(b)]$. Thus $[x \circ y] = [r < b]$, which by our assumptions equals $[x' \circ y'] = [gr < b]$. By Lemma 11, we have $[y] = [\phi(r) < \phi(b)] = [\phi(gr) < \phi(b)] = [y']$. □

6. Applications

Vassiliev introduced in his work on knot invariants ([Va93]) the complexes of disconnected graphs. The vertex set of such a complex DG_n consists of all 2-element subsets of $\{1, \ldots, n\}$, indexing all possible edges in a graph on n vertices. The simplices of DG_n are exactly the edge sets of disconnected graphs on n vertices. DG_n carries a $\mathcal{S}_n$-action induced by the action on the graph vertices, though this action does not fulfill Condition R. But there is an induced $\mathcal{S}_n$-action on the face poset $\mathcal{F}(DG_n)$, which in turn lets us explore the trisp $\Delta(\overline{\mathcal{F}(DG_n)})/\mathcal{S}_n = \mathrm{Bd}(DG_n)/\mathcal{S}_n$ with our tools, which simplifies the first analysis of this trisp by Kozlov in [Ko08b].

Let $\overline{\Pi_n}$ be the poset of partitions of $\{1, \ldots, n\}$ ordered by refinement except $1|2|\ldots|n$ and $12\ldots n$, let $\phi : \overline{\mathcal{F}(DG_n)} \to \overline{\mathcal{F}(DG_n)}$ be the map taking each graph G to its transitive closure, that is the direct sum of the complete graphs on each of the components of G. So $\phi(G)$ can be understood as a partition of $\{1, \ldots, n\}$ and $\phi(\overline{\mathcal{F}(DG_n)}$ is isomorphic to $\overline{\Pi_n}$.

Corollary 14. *The trisp* $\mathrm{Bd}(DG_n)/\mathcal{S}_n$ *is collapsible.*

Proof. Note that ϕ is a G-equivariant ascending closure operator, so by Remark 5, the prerequisites for Corollary 8 are fulfilled. Hence $\mathrm{Bd}(DG_n)/\mathcal{S}_n$ collapses onto $\Delta(\overline{\Pi_n})/\mathcal{S}_n$, the collapsibility of which has been shown in [Ko00]. □

Our results allow us to tackle the nerve of $\overline{\mathcal{F}(DG_n)}/\mathcal{S}_n$ using the same map ϕ:

Corollary 15. *The trisp $\Delta(\overline{\mathcal{F}(DG_n)}/\mathcal{S}_n)$ is collapsible.*

Proof. By Propositions 13 and 12, $\Delta(\overline{\mathcal{F}(DG_n)}/\mathcal{S}_n)$ collapses onto $\Delta(\overline{\Pi_n}/\mathcal{S}_n)$. The objects of the category $\overline{\Pi_n}/\mathcal{S}_n$ can be indexed by the nontrivial number partitions of n, between which $2+1+\ldots+1$ is minimal. We will show that this is in fact a terminal object, hence $\overline{\Pi_n}/\mathcal{S}_n$ is collapsible.

Let $\pi = \pi_1|\pi_2|\ldots \in \overline{\Pi_n}$, denote by $\overline{ab}$ the partition with one set $\{a,b\}$, rest singletons. Fix $a,b \in \pi_1$, let $\overline{cd}$ be another refinement of π. If c,d are in π_1 as well, then $g := (ac)(bd)$ stabilizes π, thus $[\overline{ab} < \pi] = [\overline{cd} < \pi]$. If c,d are in some other set π_i, then we decompose $\overline{ab} < \pi$ and $\overline{cd} < \pi$ into $\overline{ab} < \overline{ab|cd} < \pi$ and $\overline{cd} < \overline{ab|cd} < \pi$, and note that $(ac)(bd), id$ map the former to the latter. Hence we obtain $[\overline{ab} < \pi] = [\overline{cd} < \pi]$, that is, for each $[\pi] \in \overline{\Pi_n}/\mathcal{S}_n$ there is only a single morphism with source $[\pi]$ and target $2+1+\ldots+1$. □

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References

[BK05] E. Babson and D.N. Kozlov. *Group actions on posets*, J. Algebra **285** (2005), no. 2, 439-450.

[Bj96] A. Björner, *Topological methods*, in "Handbook of combinatorics", Vol. 1, 2, 1819-1872, Elsevier, Amsterdam, 1995.

[Ko00] D.N. Kozlov. *Collapsibility of $\Delta(\Pi_n)/\mathcal{S}_n$ and some related CW complexes.* Proc. Amer. Math. Soc. **128** (2000), no. 8, 2253-2259.

[Ko04] D.N. Kozlov. *A simple proof for folds on both sides in complexes of graph homomorphisms.* Proc. Amer. Math. Soc. **134** (2006), no. 5, 1265-1270 (electronic).

[Ko08a] D.N. Kozlov. *Combinatorial algebraic topology. Algorithms and Computation in Mathematics*, 21. Springer, Berlin, 2008. xx+389 pp.

[Ko08b] D.N. Kozlov. *Closure maps on regular trisps*, preprint (2008).

[Va93] V.A. Vassiliev. *Complexes of connected graphs.* The Gel´fand Mathematical Seminars, 1990-1992, 223-235, Birkhäuser Boston, Boston, MA, 1993.

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